The Jason Lin Gauntlet

MOCK EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – **2 hours**
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided at the back of this paper
- For questions in Section II, show relevant mathematical reasoning and / or calculations and/or calculations

Total marks: 120

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 110 marks (pages 6 – 16)

- Attempt Questions 11 – 17
- Allow about **1 hour and 45 minutes** for this section

Office Use Only - Do NOT write anything or make any marks below this line.

Section 2

[illegible]
[illegible] Questions 1 – 20
[illegible] about 17 minutes for this section.

Use the multiple choice answer sheet for Questions 1 – 15.

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1 How many solutions does the following equation have in $\mathbb{Z}$?

$$x^{[illegible]} + x^{[illegible]} = 0$$

A. 0
B. 1
C. 2
D. 3

2 A 3 letter password is to be made from the following keypad:

1	2	3

How many passwords are each digit at least one?

A. [illegible]
B. [illegible]
C. [illegible]
D. [illegible]

3. Suppose that numbers 1 to 10 are arranged in a circle. It is known that there will always exist k consecutive numbers with a sum of at least 28. What is the smallest possible value of k?

(A) 2
(B) 3
(C) 4
(D) 5

4. Consider the integral $\int_0^{10} f\,dx$. What value of c minimizes the integral?

(A) $c=\frac{1}{10}$
(B) $c=\frac{1}{9}$
(C) $c=\frac{1}{8}$
(D) $c=\frac{1}{7}$

5. "Regular" finds once that $\lim_{x\to 0}\frac{\sin x}{x}=\frac{0}{0}=\frac{\sin 0}{0}$ then $\lim_{x\to 0}\frac{\sin x}{x}=\lim_{x\to 0}\frac{\cos x}{1}$. Using this rule, what is the value of the following limit?

$$\lim_{x\to 0}\frac{1}{x}\int_0^x \frac{\sin t}{t}\,dt$$

(A) 0
(B) 1
(C) $\frac{\pi}{2}$
(D) $\frac{\pi}{4}$

6 [illegible]

A. [illegible]

B. [illegible]

C. [illegible]

D. [illegible]

7 [illegible]

A. [illegible]

B. [illegible]

C. [illegible]

D. [illegible]

8 [illegible]

[illegible]

A. [illegible]

B. [illegible]

C. [illegible]

D. [illegible]

9 A particle moves according to the following displacement equation.

$$x = 2\cos(nt) - \sin((2n-1)t)$$

How many rational values of n are there such that x moves in simple harmonic motion?

A. 2

B. 3

C. 4

D. 5

10 The notation a mod b refers to the remainder when a is divided by b. Using this fact, what is the value of the following limit?

$$\lim_{n\to\infty} \frac{1}{n^3} \sum_{a=1}^{n} \sum_{b=1}^{n} a \bmod b$$

A. 0

B. $\frac{1}{4}$

C. $\frac{2}{3}$

D. 1

Section B

90 marks
Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section.

Answer each question in the appropriate writing booklet. Extra writing booklets are available.

For questions in Section B, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use the Question 11 Writing Booklet.

(a) Using mathematical induction, prove that $(3n+1)\times 7^{n}-1$ is always divisible by 9 for all integers n. 3

(b) Find a possible set of equations for three skew lines such that the shortest distance between each pair of lines is 1 unit. 3

(c) Sketch the region on the Argand diagram satisfying the following inequality. 3

$$\frac{\pi}{4}\le \arg\left(\frac{z-1-i}{z+1-i}\right)\le \frac{3\pi}{4}$$

(d) Let $a, b \in \mathbb{R}$ satisfy $a+b=1$. Prove the following inequality. 3

$$a^{2}+4b\left(a-b^{2}\right)\ge \frac{3}{16}$$

Question 11 continues on page 7

[illegible]

(a) By using the substitutions $u = \sin x + \frac{1}{\sin x}$ and $v = \sin x - \frac{1}{\sin x}$, evaluate the following integral. [illegible]

$$\int \frac{\sin^3 x}{\sin^4 x + 1}\,dx$$

[illegible]

[illegible]

[illegible]

[illegible]

(a) [illegible] $y = \frac{1}{x}$ [illegible] $x = 1$ [illegible]

(b) [illegible] [2]

$$\frac{1}{n+1} < \ln\left(\frac{n+1}{n}\right) < \frac{1}{n}$$

(c) [illegible] $\frac{n}{n+1} < \ln(n!) < \frac{n}{n}$ [2]

(a) [illegible] $y = \frac{1}{x}$ [illegible] $y = \frac{1}{x^2}$ [illegible] [2]

[illegible]

End of Question 10

Please turn over

Question 10 (17 marks) Use the Question 10 Writing Booklet.

(a) Suppose z_1 and z_2 both have a modulus of 2. Prove the following inequality. 2

$$|2 + z_1 z_2| = |2 + z_1| \, |2 + z_2|$$

(b) (i) Prove that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$. 2

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \sin(x) - \cos(x) + \sin(x)\cos(x)\,dx$. 2

(c) It is given that $I_n = \int_0^{\frac{\pi}{4}} \tan^n(4x)\,dx$ for the all integer values $n \geq 2$. Prove the following inequality. 3

$$I_n + I_{n-2} = \frac{1}{4(n-1)}$$

(d) Let a and b be positive integers such that $a - b = 11$. Prove the following inequality. 3

$$a^3 - b^3 > 11a - 11b + 2ab + 121$$

(e) Using the AM/GM inequality or otherwise, solve the following equation for $x, y, z > 0$. 3

$$x^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}} + y^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}} + z^{2^{2^{2^{2^{2^{2^{2^{2^{2^{2}}}}}}}}}} = 3xyz$$

End of Question 10

Please turn over

Question [illegible] [illegible]

(a) [illegible] 3

$$[illegible]$$

(b) [illegible] 3

$$[illegible]$$

(c) (i) [illegible] 1

(ii) [illegible] 3

$$[illegible]$$

(d) [illegible]

(i) [illegible] 1

(ii) [illegible] 2

(iii) [illegible] 2

End of Question [illegible]

Please turn over

Question 15 (15 marks) Use the Question 15 Writing Booklet.

(a) Let w be a non-real fifth root of unity.

(i) Prove that w is a root of the polynomial $z^8+z^4+z^2+z+1=0$. **1**

(ii) Hence, or otherwise, factorise $z^8+z^4+z^2+z+1=0$ into two quartic factors with integer coefficients. **3**

(b) It is given that $|z|>|w|$, where z and w are complex numbers. Prove the following inequality. **3**

$$\text{Re}\left(\frac{z}{z-w}\right)>\frac{1}{2}$$

(c) (i) Let $\theta=\tan^{-1}x+\tan^{-1}y$. Prove the following. **2**

$$\tan\theta=\frac{x+y}{1-xy}$$

(ii) If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\dfrac{\pi}{2}$, show the following. **2**

$$xy+yz+zx=1$$

(iii) Let $\alpha_n=\tan^{-1}x_1+\tan^{-1}x_2+\ldots+\tan^{-1}x_n$, where $n\geq 1$. Using mathematical induction, prove the following expression, **4**

$$\tan\alpha_n=-\frac{\text{Im}(\beta_n)}{\text{Re}(\beta_n)}$$

where $\beta_n=(1-ix_1)(1-ix_2)\ldots(1-ix_n)$

End of Question 15

Please turn over

[illegible]

[illegible]

(a) [illegible]

(i) [illegible] 1

(ii) [illegible] 2

(iii) [illegible] 3

(iv) [illegible] 3

[illegible]

[illegible]

[illegible]

(b) Let $F_1, F_2, \ldots, F_n$ be the sequence of Fibonacci numbers, where $F_{n-1} + F_n = F_{n+1}$ for all natural numbers n. It is known that $F_1 = F_2 = 1$.

(i) Using mathematical induction with two inductive hypotheses, show that both $F_{n-1}^2 + F_n^2 = F_{2n-1}$ and $F_{n-1}^2 + 2F_{n-1}F_n = F_{2n-1}$ for all integers $n \geq 1$. 4

(ii) Hence, prove that F_{n-1}, F_{n+1} and F_{2n-1} form a Pythagorean triple. 2

[illegible]

[illegible]

[illegible]

(a) Let n be an integer such that $n \geq 2$ and let r be the n^{th} root of unity with the smallest positive argument. Define $s_k = \dfrac{1 + r + r^2 + \cdots + r^k}{1 + r^k}$ for $k = 1, 2, \ldots, n-1$.

(i) Show that $s_k = \cot\left(\dfrac{\pi k}{2n}\right)$. [2]

(ii) Prove the following identity. [3]

$$\sum_{k=1}^{n-1} \left(1 + \cot^2\left(\frac{\pi k}{2n}\right)\right) = n \cot\left(\frac{\pi}{2n}\right) \cot\left(\frac{(n-1)\pi}{2n}\right)$$

(iii) From here onwards, let n be such that it is an odd integer with $n \geq 3$. Show the following. [3]

$$\sum_{k=1}^{n-1} \left(1 + \cot^2\left(\frac{\pi k}{2n}\right)\right) = 1 + n^2$$

(iv) Hence, show the following. [2]

$$1 + \cot^2\left(\frac{\pi}{2n}\right) + \cot^2\left(\frac{2\pi}{2n}\right) + \cdots + \cot^2\left(\frac{(n-1)\pi}{2n}\right) = n^2$$

(v) Prove the following statement. [2]

$$\sum_{k=1}^{n-1} \cot^2\left(\frac{\pi k}{2n}\right) = n^2 - 1$$

(vi) Deduce the following. [2]

$$\sum_{k=1}^{n-1} \cot^2\left(\frac{\pi k}{2n}\right) = n^2$$

(vii) Define $S = \sum_{k=1}^{n-1} \frac{1}{\sin^2\left(\frac{\pi k}{2n}\right)}$. By using parts (i) and (vi), prove the following. [1]

$$S = \text{[illegible]} - n$$

[illegible] continues on page 16

[illegible]

[illegible] 4

[illegible]

[illegible]

[illegible]

[illegible]